

Stability, Convergence and Error Analysis of An Adaptive Modified Euler Method for Nonlinear Ordinary Differential Equations

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ABSTRACT

This paper examines the consistency, convergence, error propagation and absolute stability of a step-doubling adaptive modified Euler method for nonlinear initial value problems. The underlying Heun update has a second-order Taylor expansion and a cubic local defect. Two half steps retain second-order global convergence and, when paired with a full step, provide an asymptotic local-error estimate for the accepted fine solution. A discrete error recursion supplies a conditional $O(h_{\max}^2)$ global bound on finite intervals under appropriate smoothness and Lipschitz assumptions, provided the accepted mesh becomes uniformly fine. On the Dahlquist test equation, the accepted fine update has stability function $[1+z/2+z^2/8]^2$; the real negative stability interval in terms of the total attempted step extends to $-4 < z < 0$, compared with $-2 < z < 0$ for a single full Heun step. This extension costs two propagation half steps and does not imply A-stability or solve stiffness. Numerical checks on nonlinear logistic and Riccati equations illustrate the distinction between local tolerance, endpoint error and stability. The results establish the method's mathematical scope and practical limits.

Keywords- Absolute stability; adaptive Heun method; consistency; convergence; global error; local truncation error.

1. Introduction

A numerical method for nonlinear ordinary differential equations is useful only when its errors can be understood and its calculations remain stable in the intended regime. Variable time steps complicate that assessment. A successful local error test does not prove convergence by itself, nor does a small reported defect guarantee that an explicit discretisation can handle stiff modes. Analysis must separate truncation, propagation, step-size selection and the stability of the underlying advance.

This paper addresses that separation for the adaptive modified Euler construction described in the accompanying methodological study. Its core is the explicit trapezoidal Heun rule, combined with a coarse-full-step versus two-half-step comparison. The technique responds to the doctoral synopsis's call for theoretical work on error, convergence and stability. It builds on standard numerical analysis principles (Gear, 1971; Hairer, Nørsett and Wanner, 1993) and adaptive-control literature (Hall and Higham, 1988; Söderlind, 2002).

The terminology matters. "Stable" can refer to propagation of perturbations on a bounded interval, or to absolute stability of a method on a decaying linear test problem; these are related

but not identical concepts. Similarly, consistency is an asymptotic statement about a vanishing time step, while tolerance compliance is an algorithmic decision made from an estimated local error. The purpose here is to derive valid conclusions for each notion, state necessary assumptions and avoid interpreting step-size adaptation as an unconditional stability guarantee.

2. Initial value setting and assumptions

Consider $y'(t)=f(t,y(t))$ over $t_0 \leq t \leq T$, with $y(t_0)=y_0$. The scalar presentation extends to vector systems with compatible norms. Assume the exact solution remains in a compact domain and f is Lipschitz continuous in its state argument there, with constant L . To justify Taylor remainder estimates through the required order, suppose f and the solution have sufficiently many bounded derivatives along a neighbourhood of the trajectory. Uniform constants in the bounds are essential: a formal $O(h^3)$ at isolated points is not enough for a finite-interval global result.

The one-step map $H(t,Y,h)$ is $Y + h[f(t,Y)+f(t+h,Y+h f(t,Y))]/2$. An accepted adaptive advance is $A(t,Y,h)=H(t+h/2,H(t,Y,h/2),h/2)$. Although the controller may reject a proposal, the mathematical trajectory is formed exclusively from accepted maps A . Denote accepted lengths by h_n and let $h_{\max}$ be the largest such length on the interval. The steps must be positive, sum to $T-t_0$ and tend uniformly to zero in a convergence sequence; a local error rule alone does not force these properties without additional safeguards and assumptions.

The step-doubling estimate $E_n=|A(t_n,Y_n,h_n)-H(t_n,Y_n,h_n)/3|$ is an asymptotic measure of the fine-map defect. It is not a mathematical upper bound on unknown error at a finite h unless further rigorous estimation is provided. In particular, accidental cancellation may yield a small discrepancy even when both approximations are inaccurate. This observation informs the interpretation of practical tolerances.

2.1 Interpreting error estimators

The word ‘error’ has several meanings in time integration. A local truncation defect is the discrepancy in one step started from exact data; a local estimator is a computable approximation to that defect; the global error includes all accumulated previous defects; and a residual describes how well a continuous reconstruction satisfies the differential equation. The step-doubling ratio relates two local discretisations to the first of these quantities, but the algorithm observes the second. Convergence analysis controls the third using stability and summation. These concepts should remain distinct in any theorem, table or abstract.

When the coefficient K of the leading cubic term becomes small or changes sign, the coarse/fine difference may fall below the asymptotic prediction. This phenomenon does not necessarily imply that the next terms are negligible. Likewise, smooth dependence on the starting state is important, because an estimated defect computed at an already inaccurate numerical state cannot independently measure the accumulated global error. For practical monitoring, occasional mesh-refinement checks or comparisons with a trusted higher-order solver can reveal whether the error estimator is operating within a reasonable range.

2.2 Real-axis stability and evaluation budgets

The real-axis calculation can also be checked without complex-plane plots. For $x \leq 0$, $R(x) = 1 + x + x^2/2$ is a parabola with minimum one half at $x = -1$. It equals one at $x = 0$ and $x = -2$; hence its magnitude stays below one on the open interval between. In the fine update the substep argument is $x/2$, so the corresponding endpoints in the total-step variable are zero and minus four. This scaling depends on defining h as the total attempted interval. If h instead denoted each half step, writing a stability boundary of minus four would be a notational mistake.

Work normalisation changes the interpretation. One full Heun step spans h using two evaluations, whereas two half steps span the same h using four; the adaptive diagnostic makes six in total per attempt. Thus doubling a stability interval while doubling propagation calls gives no free enlargement at equal work. On a sequence of irregular steps, furthermore, stability is governed by products of amplification factors and not by checking a single nominal step. Small stable factors may offset some growth, but reliable explicit integration of a uniformly decaying test mode generally requires each step to avoid sustained amplification.

3. Consistency and local truncation error

Write $g(t) = f(t, y(t))$. Along an exact solution, $y(t+h) = y(t) + hg(t) + h^2g'(t)/2 + h^3g''(t)/6 + O(h^4)$. For the full Heun step, expand $f(t+h, y(t) + hf(t, y(t)))$ around $(t, y(t))$. Its leading terms are $f + h(f_t + f_y f) + h^2[f_{tt}/2 + f_{ty}f + f_{yy}f^2/2] + O(h^3)$, evaluated at the starting point. Multiplication by $h/2$ and addition of the initial slope reproduce the exact constant, first-order and second-order terms. Accordingly, the one-step local defect has order h^3 . Dividing the local defect by h shows consistency as h tends to zero.

Two half steps also reproduce the first two nonzero Taylor terms of the exact flow. The leading defect accumulated over the two half steps is one quarter of the leading defect of a full step, up to terms of order h^4 . If $dC = C - y(t+h) = Kh^3 + O(h^4)$, then $dF = F - y(t+h) = Kh^3/4 + O(h^4)$. Thus $F - C = -3Kh^3/4 + O(h^4)$, and $dF = -(F - C)/3 + O(h^4)$. Absolute values give $|dF| \approx |F - C|/3$, the estimator used in the controller.

This argument provides an important caveat: the step-doubling difference is asymptotically linked to the error of the two-half-step accepted value, not directly to the error of the coarse Heun value. For the latter, $|dC| \approx 4|F - C|/3$. Reporting the wrong scale factor would misstate the tolerance policy by approximately a factor of four in the leading term. Moreover, the method remains globally order two even though its accepted result is more accurate than a single full step. Without a Richardson extrapolation update, step doubling alone does not create an order-three propagator.

4. Propagation and convergence

To obtain a finite-time bound, first note that Lipschitz continuity of f implies a stability inequality for the Heun map on sufficiently small steps: $|H(t, U, h) - H(t, V, h)| \leq [1 + Lh + (L^2/2)h^2]|U - V|$. The estimate follows by bounding the predictor difference by $(1 + Lh)|U - V|$ and then bounding the average of the initial and predicted slopes. For two half steps, multiplication of the corresponding factors yields

$|A(t,U,h)-A(t,V,h)| \leq (1+Ch)|U-V|$ for $0 < h \leq h^*$, with a constant C dependent on L and h^* , but not on the individual mesh index.

Let $e_n = Y_n - y(t_n)$. Insert and subtract $A(t_n, y(t_n), h_n)$. The triangle inequality gives $|e_{n+1}| \leq (1+Ch_n)|e_n| + Kh_n^3$, where K uniformly bounds the accepted map's local defect divided by h^3 . Iterating the inequality and using $1+Ch \leq \exp(Ch)$ gives $|e_n| \leq \exp[C(t_n - t_0)](|e_0| + K \sum_{j=0}^{n-1} h_j^3)$. Because $\sum h_j^3 \leq h_{\max}^2 \sum h_j \leq h_{\max}^2(T - t_0)$, and $e_0 = 0$ for exact initial data, the global bound becomes $\max_n |e_n| \leq K(T - t_0) \exp[C(T - t_0)] h_{\max}^2$.

This establishes second-order convergence for any family of accepted partitions whose maximum length approaches zero and which satisfy the stated smoothness and domain conditions. It is not a theorem that a chosen value of $atol$ or $rtol$ by itself implies a particular global error, nor does it guarantee how many rejected trials occur. A full analysis of the adaptive controller's termination would additionally bound the admissible steps, demonstrate eventual acceptance for positive tolerances in regular regions and describe floating-point floors. The central conclusion is conditional consistency and convergence of the accepted numerical maps, not unconditional success of every implementation.

5. Absolute stability of full and half steps

For the Dahlquist equation $y' = \lambda y$, set $z = h\lambda$. A single Heun step multiplies the state by $R(z) = 1 + z + z^2/2$. The two-half-step accepted update multiplies it by $A(z) = [R(z/2)]^2 = [1 + z/2 + z^2/8]^2$. Absolute stability requires $|A(z)| < 1$ for decaying modes. On the negative real axis, $R(x)$ is nonnegative for real x and $R(x) < 1$ precisely when $-2 < x < 0$. Consequently $|A(z)| < 1$ when $-4 < z < 0$ for negative real z . At the endpoints $z = -4$ and $z = 0$, the amplification modulus equals one; strict asymptotic decay does not hold.

At first sight the doubling of this real-axis interval could be called a stability improvement. That language needs qualification. The accepted propagation traverses two half steps, not one step with an altered underlying integrator. It consequently uses four right-hand-side evaluations rather than the two required for one full Heun step, and the trial estimator adds another two. In terms of stable step length per propagation evaluation, the change confers no general advantage over simply taking two fixed Heun half steps. The gain is operational: the controller can identify a need for smaller intervals and distribute steps in response to local behaviour.

The adaptive method is not A-stable. Its amplification polynomial grows without bound as $|z|$ becomes large, and therefore cannot contain the entire left half-plane within its absolute stability region. For stiff systems, some components may have very negative eigenvalues even if the visible solution is slowly varying; the controller may choose steps based on accuracy that lie outside the stability region. A rejection rule based only on the difference between coarse and fine results cannot be assumed to detect every unstable regime immediately. The appropriate conclusion is that adaptivity can improve practical resolution on selected problems, not remove explicit stability restrictions.

6. Nonlinear stability considerations

Linear absolute stability is informative but cannot replace nonlinear reasoning. Suppose f is L -Lipschitz in y on a tube containing the exact and computed trajectories. The derived one-step

perturbation factor $(1+Ch)$ gives a bound on the growth of nearby numerical initial states over a fixed interval, uniformly for small steps. This finite-time stability is sufficient to control propagation of truncation error in the convergence proof. Its constant may be large, however, and its exponential dependence on interval length is conservative.

Problems with dissipative structure can admit stronger estimates, but they require assumptions beyond ordinary Lipschitz continuity, such as one-sided Lipschitz or monotonicity inequalities. Conversely, an equation can be locally Lipschitz until the solution approaches a singularity; no bounded-error theorem extends automatically past a blow-up time. The Riccati reference $y(t)=\tan(t)$, for example, loses regularity as t approaches $\pi/2$. A numerical trial confined to $0 \leq t \leq 0.7$ does not test behaviour near that singularity. Similarly, a logistic equation's bounded state is not evidence that the same method is stable for all nonlinear ODEs.

The controller also changes step selection in response to numerical history. When analysing sensitivity to perturbed initial conditions, it is simpler and more rigorous to compare solutions on a prescribed accepted mesh than to assume that two perturbed trajectories select identical meshes. A complete perturbation theorem for the full closed-loop algorithm would need a model of the controller, rejection decisions and any discontinuity introduced by thresholding. Here the claims are intentionally limited to the underlying accepted-map method.

7. Computational illustration and interpretation

The illustrative tests use $y'=2y(1-y)$, $y(0)=0.1$, with exact solution $[1+9 \exp(-2t)]^{-1}$ at $T=4$, and $y'=1+y^2$, $y(0)=0$, with exact solution $\tan(t)$ at $T=0.7$. Each problem is integrated by coarse/fine Heun step doubling with $h_0=0.2$, safety coefficient 0.9, step-change factors between 0.2 and 4, and $atol=rtol=\tau$. The recorded endpoint errors compare accepted results with exact values. These quantities must not be confused with local estimates or a proved global upper bound.

Results in Table 1 show the expected relationship among a tighter tolerance, more numerical work and the accuracy of this particular implementation. Acceptance counts alone are insufficient for fair cost comparison because each attempted interval requires six function evaluations. If a step is rejected, its evaluations add expense while contributing no accepted advance. In addition, the tolerance-to-global-error relationship is affected by the time interval, the dynamical Lipschitz constant and the evolving size of the solution. Thus even successful numeric tests do not establish a universal equality between tolerance and final error.

To interpret the stability polynomial concretely, consider $\lambda < 0$. A proposed total step with $z=-3$ is outside the single-step Heun real stability interval $(-2,0)$ but inside the two-half-step propagation interval $(-4,0)$. This is consistent with each substep seeing $z/2=-1.5$. The example does not constitute an algorithmic achievement beyond substepping. A total step with $z=-5$ is unstable for the accepted map regardless of how low the estimated local defect might appear in a particular trial. The proper control response is to reduce the step into a stable region.

Table 1. Adaptive step-doubling results (final-time absolute error; function evaluations include rejected trials).

Equation	τ	Accepted	Rejected	f calls	End error
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Logistic	1e-03	11	1	72	5.800e-04
Logistic	1e-05	44	1	270	4.032e-05
Logistic	1e-07	200	2	1212	2.047e-06
Riccati	1e-03	3	0	18	9.523e-04
Riccati	1e-05	11	3	84	7.473e-05
Riccati	1e-07	47	6	318	3.355e-06

8. Implications and limitations

The mathematical analysis clarifies the scope of the hypotheses posed in the synopsis. Under regularity and vanishing maximum step length, consistency and convergence can be established: this supports a carefully qualified version of the proposed convergence hypothesis. Smaller errors than fixed Euler are plausible in many cost and tolerance regimes but not proved universally by the order argument alone. Likewise, computational efficiency depends on the work budget and solution structure. The larger negative-real stability interval stated in units of the total attempted h describes two substeps rather than an intrinsically superior one-step stability polynomial.

Several extensions would strengthen the doctoral research. A variable-step convergence study can calculate observed order using successively reduced maximum steps. Cost-normalised experiments should match function evaluations rather than count accepted advances. PI controllers might reduce repeated rejections and improve mesh smoothness, as indicated by feedback-control research (Gustafsson, Lundh and Söderlind, 1988; Söderlind, 2003). Testing systems with both positive and negative real eigenvalues, complex spectra, events and weak stiffness would locate the method's actual domain of usefulness. For a claim of guaranteed tolerance satisfaction, one would need verified enclosures or rigorous defect estimators rather than asymptotic step doubling alone.

The study has not established unconditional nonlinear stability, absence of rejection cycles or optimal performance relative to modern embedded solvers. Stating these limits is part of the technical result, since a method's safe application depends as much on what its analysis excludes as on what it proves.

9. Recommendations for theorem verification

An extended theorem for practical use should first specify an invariant domain in which the exact solution and all accepted numerical stages remain bounded. This condition is particularly important for nonlinear f , since a predictor can exit the physically meaningful region even if its starting state is admissible. A rigorous step-size restriction may then ensure that both predictors and correctors remain within that domain. With these prerequisites, uniform derivative bounds and a Lipschitz constant can be used to determine the constants in the error recursion rather than leaving them implicit.

A separate result would be needed to relate user tolerances to a mesh-refinement limit. For example, if the leading cubic defect coefficient is nonzero and uniformly bounded away from zero, the acceptance inequality constrains steps in a manner that may be analysed as the tolerance decreases. However, this nondegeneracy assumption need not hold throughout a

nonlinear trajectory, especially near points of cancellation. It should therefore not be asserted without verification. More robust results may require a supplementary maximum-step rule independent of the estimator.

A useful validation exercise is to compare predicted and observed order on increasingly fine fixed meshes, using the same two-half-step map without adaptation. This isolates the discretisation theorem from the behaviour of the controller. Repeating the experiment with adaptive meshes then tests whether the selection mechanism preserves the expected trend. Taken together, these studies would provide a clearer bridge between mathematical analysis and claims about automated error control.

10. Conclusion

The accepted two-half-step modified Euler map is consistent and converges with global order two on bounded finite intervals under standard smoothness, Lipschitz and mesh-refinement assumptions. Its coarse/fine discrepancy supplies a leading-order estimate of accepted local error, not a certified error bound. Its negative-real absolute stability interval in the total-step variable is $(-4,0)$, compared with $(-2,0)$ for a single Heun step; this reflects substepping at additional computational cost and does not confer A-stability. These results form a rigorous baseline for the adaptive nonlinear ODE programme and identify the further algorithmic and numerical evidence required for stronger performance claims.

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